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EXTENSION OF NESTED ARRAYS WITH THE FOURTH-ORDER DIFFERENCE CO-ARRAY ENHANCEMENT

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ABSTRACT

To reach a higher number of degrees of freedom by exploiting the fourth-order difference co-array concept, an effective structure extension based on two-level nested arrays is proposed. It increases the number of consecutive lags in the fourth-order difference co-array, and a virtual uniform linear array (ULA) with more sensors and a larger aperture is then generated from the proposed structure, leading to a much higher number of distinguishable sources with a higher accuracy. Compressive sensing based approach is applied for direction-of-arrival (DOA) estimation by vectorizing the fourth-order cumulant matrix of the array, assuming non-Gaussian impinging signals.

***Index Terms*—** Fourth order, difference co-array, cumulant, sparse array, direction of arrival estimation, compressive sensing.

1. INTRODUCTION

Co-array equivalence plays an important role in designing sparse array structures [1, 2], leading to an effective solution for underdetermined direction-of-arrival (DOA) estimation. One class of arrays employing this concept is the co-prime array [3], where both the spatial smoothing based subspace methods [3–5] and compressive sensing (CS) based methods [6–10] can be used for DOA estimation. Another class of arrays falling into this category is the nested array [11], and spatial smoothing based subspace approaches have been employed for DOA estimation [11–13].

Most of the work about DOA estimation for the aforementioned structures are based on the second-order difference co-array concept. Actually, high-order statistics have been exploited for DOA estimation over the decades to resolve more sources than the number of sensors. The virtual array concept for the fourth-order cumulants based DOA estimation [14, 15] is presented in [16]. Based on the $2q$-th order cumulants [17, 18], the $2q$-th order difference co-array concept is proposed in [19]. Then, $2q$-level nested arrays are proposed with a substantial increase in the number of degrees of freedom (DOFs) [19], and spatial smoothing based subspace method is applied to find the DOAs. However, although the $2q$-level nested array provides a systematic way for convenient structure construction, it is not optimum and further improvement is possible since the physical array aperture and the symmetric features in the high-order difference co-array have not been fully exploited in array construction.

In this paper, we focus on how to more effectively construct an array based on the fourth-order difference co-array concept, and a sparse array extension based on the standard two-level nested array is proposed. It is shown that the number of DOFs of the new construction is always larger than the standard two-level nested array, and when the total number of physical sensors is less than 21, the proposed structure will always give more DOFs than the existing four-level nested array, while for 20 physical sensors for our proposed structure, the number of virtual ULA sensors at the fourth-order difference co-array stage can be 2223, which is sufficient for most applications. With this significantly increased DOFs, CS-based method is employed for DOA estimation.

This paper is organized as follows. A review of DOA estimation based on the four-level nested array is presented in Sec. 2. The specifically designed array structure based on two-level nested arrays is proposed in Sec. 3. Simulation results are provided in Sec. 4, and conclusions are drawn in Sec. 5.

2. REVIEW OF DOA ESTIMATION BASED ON THE FOUR-LEVEL NESTED ARRAY

Generally, we use S to represent the set of sensor positions, and an N-sensor linear array can be expressed as

$$S = \{p_0 \cdot d, p_1 \cdot d, \dots, p_{N-1} \cdot d\} , \tag{1}$$

where $p_n \cdot d$ is the position of the n-th sensor, $n = 0, 1, \dots, N-1$, and d is the unit spacing.

Assume that there are K mutually uncorrelated far-field narrowband signals $s_k(t)$ impinging from incident angles $\theta_k, k = 1, 2, \dots, K$, respectively. After sampling with a frequency f_s, the array output model in discrete form is given by

$$\mathbf{x}[i] = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}[i] + \overline{\mathbf{n}}[i] , \tag{2}$$

where $\mathbf{x}[i]$ is the observed discrete signal vector, the source signal vector $\mathbf{s}[i] = [s_1[i], \dots, s_K[i]]$, and $\overline{\mathbf{n}}[i]$ is the noise vector. The steering matrix $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_K)]$, with its k-th column vector $\mathbf{a}(\theta_k)$, i.e. the steering vector corresponding to the k-th source signal, expressed as

$$\mathbf{a}(\theta_k) = \left[e^{-j\frac{2\pi p_0 d}{\lambda}\sin(\theta_k)}, \dots, e^{-j\frac{2\pi p_{N-1} d}{\lambda}\sin(\theta_k)} \right]^T . \tag{3}$$

As a special array structure exploring the fourth-order difference co-array, the Four-Level Nested Arrays (FL-NA) proposed in [11,19] has four sub-arrays. With $N_0 = 0$, we have $N_0 + 1 = 1$. For $1 \le m \le 3$, the m-th sub-array has N_m sensors located at

$$\left\{nd\left[\prod_{\tilde{m}=0}^{m-1}(N_{\tilde{m}}+1)\right], n = 1,2,\ldots,N_m\right\} , \quad (4)$$

while the sensors of the fourth sub-array with $N_4 + 1$ sensors are located at

$$\left\{nd\left[\prod_{\tilde{m}=0}^{3}(N_{\tilde{m}}+1)\right], n = 1,2,\ldots,N_4+1\right\} . \quad (5)$$

Then, there are $N = \sum_{m=1}^{4} N_m + 1$ physical sensors in total.

Under the assumption of Gaussian white noise, the fourth-order cumulant matrix of the observed column vector $\mathbf{x}[i]$ for the arrangement indexed by l can be obtained by

$$\mathbf{C}_{4,\mathbf{x}}(l) = \sum_{k=1}^{K} c_{4,s_k}\left[\mathbf{a}(\theta_k)^{\otimes l} \otimes \mathbf{a}(\theta_k)^{*\otimes(2-l)}\right] \times \left[\mathbf{a}(\theta_k)^{\otimes l} \otimes \mathbf{a}(l,\theta_k)^{*\otimes(2-l)}\right]^H , \quad (6)$$

where $l = 0, 1$. $\mathbf{a}(\theta_k)^{\otimes l}$ denotes $\mathbf{a}(\theta_k)\otimes\ldots\otimes\mathbf{a}(\theta_k)$ with $\mathbf{a}(\theta_k)$ for l times, and $\{\cdot\}^*$ represents the conjugate operation. The fourth-order auto-cumulant of source signal $s_k[i]$ can be expressed as

$$c_{4,s_k} = \text{Cum}\left\{s_k[i], s_k[i], s_k^*[i], s_k^*[i]\right\} , \quad (7)$$

where $1 \le k \le K$, and $\text{Cum}\{\cdot\}$ is the cumulants operator.

We set $l = 1$, and by vectorizing $\mathbf{C}_{4,\mathbf{x}}(1)$ we obtain

$$\mathbf{z} = \text{vec}\left\{\mathbf{C}_{4,\mathbf{x}}(1)\right\} = \mathbf{B}\mathbf{u} . \quad (8)$$

Equation (8) characterises a virtual array, whose equivalent steering matrix $\mathbf{B} = [\mathbf{b}(\theta_1), \ldots, \mathbf{b}(\theta_K)]$ with each column vector $\mathbf{b}(\theta_k) = [\mathbf{a}(\theta_k) \otimes \mathbf{a}(\theta_k)^*]^* \otimes [\mathbf{a}(\theta_k) \otimes \mathbf{a}(\theta_k)^*]$. The equivalent signal vector $\mathbf{u} = [c_{4,s_1}, c_{4,s_2}, \ldots, c_{4,s_K}]$.

To obtain the DOA results, subspace methods can be applied directly to $\mathbf{C}_{4,\mathbf{x}}(l)$ in (6), and spatial smoothing based subspace methods can be employed in the virtual model characterised by (8).

3. SPARSE ARRAY EXTENSION BASED ON THE FOURTH-ORDER DIFFERENCE CO-ARRAY CONCEPT

3.1. The fourth-order difference co-array perspective for a two-level nested array

For a given physical array in (1), the *second-order difference co-array* (also known as *difference co-array*) set is defined as

$$\mathbb{C}_{\mathbb{A}} = \Phi_A \cdot d , \quad (9)$$

with the set of difference co-array lags Φ_A given by

$$\Phi_A = \{p_{n_1} - p_{n_2}\} , \quad (10)$$

where $0 \le n_1, n_2 \le N - 1$.

According to [19], the set of *fourth-order difference co-array* is defined as

$$\mathbb{C}_{\mathbb{B}} = \Phi_B \cdot d , \quad (11)$$

with the set of the fourth-order difference co-array lags

$$\Phi_B = \{p_{n_1} + p_{n_2} - p_{n_3} - p_{n_4}\} . \quad (12)$$

where $0 \le n_1, n_2, n_3, n_4 \le N - 1$.

For a given unit spacing d, a general Two-Level Nested Array (TL-NA) consists of two sub-arrays [11], where the first sub-array has N_1 sensors starting from the position $1d$ with d as the spacing between adjacent physical sensors, and the second sub-array has N_2 sensors starting from the position $(N_1 + 1)d$ with an inter-element spacing $(N_1 + 1)d$.

There are $N_1 + N_2$ physical sensors in total, and the difference co-array achieved in the set of co-array lags Φ_A can be expressed as

$$\Phi_A = \{\mu, -N_2(N_1+1)+1 \le \mu \le N_2(N_1+1)-1\} . \quad (13)$$

Φ_A only contains consecutive integers from $-N_2(N_1 + 1) + 1$ to $N_2(N_1 + 1) - 1$, corresponding to a ULA of $2N_2(N_1 + 1) - 1$ virtual sensors. The set Φ_B in (12) can be rewritten as

$$\Phi_B = \{(p_{n_1} - p_{n_3}) - (p_{n_4} - p_{n_2})\} . \quad (14)$$

Note that $(p_{n_1} - p_{n_3}) \in \Phi_A$ and $(p_{n_4} - p_{n_2}) \in \Phi_A$. Then the fourth-order difference co-array set Φ_B for the TL-NA is given by

$$\Phi_B = \{\mu, -2N_2(N_1+1)+2 \le \mu \le 2N_2(N_1+1)-2\} . \quad (15)$$

The number of consecutive integers is increased to $4N_2(N_1 + 1) - 3$ in Φ_B, which suggest that more DOFs can be exploited for DOA estimation by employing the fourth-order difference co-array based method. However, the set Φ_A of the TL-NA indicates that the virtual array generated at the difference co-array stage is only a ULA, and the increase in the number of consecutive integers from Φ_A to Φ_B is limited.

3.2. Sparse array extension with the fourth-order difference co-array enhancement

To fully exploit the advantages of the fourth-order difference co-array, a novel Sparse Array extension with the Fourth-Order difference co-array Enhancement based on the TL-NA (SAFOE-NA) is proposed, optimising the consecutive integers at the fourth-order difference co-array stage with each introduced physical sensor of the third sub-array.

Define the sensor positions of the introduced third sub-array as $\alpha_{n_3}d$, $0 \le n_3 \le N_3 - 1$, where N_3 is the sensor number of the third sub-array. Since the co-array lags at each order are symmetric about 0, our analysis only takes the positive part into consideration. In the set Φ_A, except for the self-difference co-array of the third sub-array, the minimum and the maximum positive integers for the cross-difference co-array associated with the n_3-th sensor at position $\alpha_{n_3}d$ can be expressed as $\alpha_{n_3} - N_2(N_1 + 1)$ and $\alpha_{n_3} - 1$, respectively. It is noted that the difference between the mentioned minimum and maximum positive integers is the physical array aperture.

According to (13), the range of consecutive integers at the fourth-order difference co-array stage associated with the n_3-th sensor can be obtained, given in the set $\phi_{\alpha_{n_3}}$

$$\phi_{\alpha_{n_3}} = \{\mu, \nu_{n_3} \le \mu \le \zeta_{n_3}\} , \quad (16)$$

where

$$\begin{aligned} \nu_{n_3} &= \alpha_{n_3} - 2N_2(N_1+1)+1 \ , \\ \zeta_{n_3} &= \alpha_{n_3} + N_2(N_1+1)-2 \ . \end{aligned} \tag{17}$$

For the starting position $\alpha_0 d$, in order to ensure the covered range by the starting position to be adjacent to the fourth-order difference co-array range of the standard TL-NA structure, the lower bound ν_0 should be the maximum integer in (15) plus 1, given as

$$\begin{aligned} \nu_0 &= \alpha_0 - 2N_2(N_1+1)+1 \\ &= 2N_2(N_1+1)-2+1 \ , \end{aligned} \tag{18}$$

and then we obtain the starting position of the third sub-array

$$\alpha_0 = 4N_2(N_1+1)-2 \ . \tag{19}$$

For the remaining sensors in the third sub-array, to maximise the number of consecutive co-array lags, the covered ranges $\phi_{\alpha_{n_3}}$, $n_3 = 0, 1, \ldots, N_3-1$, should be adjacent to each other, expressed as

$$\nu_{n_3} = \zeta_{n_3-1}+1, 1 \leq n_3 \leq N_3-1 \ . \tag{20}$$

Then the inter-element spacing is obtained by

$$\alpha_{n_3} - \alpha_{n_3-1} = 3N_2(N_1+1)-2 \ . \tag{21}$$

According to (19) and (21), the third sub-array is also a uniform linear sub-array with the starting position of $\big[4N_2(N_1+1)-2\big]d$ and the inter-element spacing $\big[3N_2(N_1+1)-2\big]d$. The maximum integer of the fourth-order difference co-array lag $\nu_{N_3-1} = (3N_3+2)N_2(N_1+1)-2N_3-2$. Finally, we have designed a sparse array structure by extending the TL-NA, with the set of the fourth-order difference co-array lags Φ_B updated to

$$\Phi_B = \{\mu, -M_0 \leq \mu \leq M_0\} \ , \tag{22}$$

where $M_0 = (3N_3+2)N_2(N_1+1)-2N_3-2$, and the number of consecutive lags is $2M_0+1$.

The inter-element spacing of the third sub-array in our proposed SAFOE-NA is $\big[3N_2(N_1+1)-2\big]d$. This larger inter-element spacing is due to exploration of the physical aperture and the symmetric information at the second-order difference co-array stage, which is not exploited in the design of the FL-NA in [19]. In fact, this inter-element spacing $\big[3N_2(N_1+1)-2\big]d$ can be considered as the original physical aperture $N_2(N_1+1)-1$ plus the number of consecutive lags at the difference co-array stage $2N_2(N_1+1)-1$.

3.3. Comparison between different structures

For a FL-NA with $N = \sum_{m=1}^{4} N_m+1$ physical sensors, the number of consecutive lags at the fourth-order difference co-array stage is roughly [19]

$$2\prod_{m=1}^{4}(N_m+1)-1 \ . \tag{23}$$

By applying the Arithmetic Mean-Geometric Mean (AM-GM) inequality, the maximum value in (23) is achieved when $N_m = \frac{N-1}{4}$, $1 \leq m \leq 4$. Then, (23) can be updated to

$$2\left(\tfrac{N+3}{4}\right)^4 - 1 \ . \tag{24}$$

Table 1. Comparison of the Fourth-Order Difference Co-Array Lags

Structures	Number of Sensors	Number of Consecutive Lags
TL-NA	N_1+N_2	$4N_2(N_1+1)-3$
FL-NA	$N=\sum_{m=1}^{4} N_m+1$	$2\prod_{m=1}^{4}(N_m+1)-1$
SAFOE-NA	$N=\sum_{m=1}^{3} N_m$	$2M_0+1^\dagger$

Array Structures	(N_1, N_2), (N_1, N_2, N_3, N_4) or (N_1, N_2, N_3)	Number of Sensors	Number of Consecutive Lags
TL-NA	(2,3)	5	33
FL-NA	(1,1,1,1)	5	31
SAFOE-NA	(1,2,2)	5	53
TL-NA	(8,9)	17	321
FL-NA	(4,4,4,4)	17	1249
SAFOE-NA	(5,6,6)	17	1413
TL-NA	(10,11)	21	481
FL-NA	(5,5,5,5)	21	2591
SAFOE-NA	(7,7,7)	21	2545

† $M_0 = (3N_3+2)N_2(N_1+1)-2N_3-2$.

In our proposed SAFOE-NA, the number of consecutive lags at the fourth-order difference co-array stage is $2M_0+1$, where

$$\begin{aligned} M_0 &= (3N_3+2)N_2(N_1+1)-2N_3-2 \\ &= 3(N_3+\tfrac{2}{3})N_2(N_1+1)-2N_3-2 \ . \end{aligned} \tag{25}$$

Since $N_m \geq 1$, $1 \leq m \leq 3$, the second term $2N_3$ is much smaller than the first term $(3N_3+2)N_2(N_1+1)$ in (25), especially when N_m becomes larger. For a simple comparison, we consider maximising the first term in M_0 to achieve the maximum number of consecutive lags. By applying the AM-GM inequality, the maximum value $M_{\max}$ is obtained when $(N_3+\frac{2}{3}) = N_2 = (N_1+1)$, with

$$M_{\max} = 3\left(\frac{N+\frac{5}{3}}{3}\right)^3 - 2\left(\frac{N}{3}-\frac{1}{9}\right) - 2 \ . \tag{26}$$

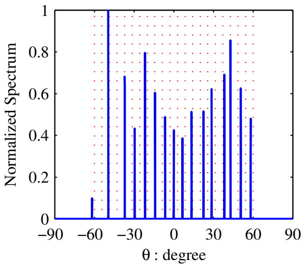
Note that all N_m, $1 \leq m \leq 4$, should be real positive integers in practice. Equations (24) and (26) are only used to compare the potential maximum values with respect to N. We compare the maximum consecutive lags by solving the following formulation

$$2\left(\tfrac{N+3}{4}\right)^4 - 1 - (2M_{\max}+1) \leq 0 \ . \tag{27}$$

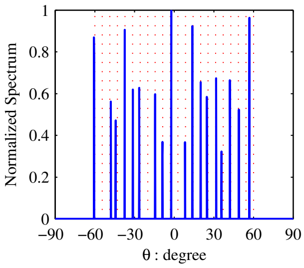
The solution to (27) corresponds to the range of sensor numbers in which more DOFs can be provided by our proposed structure than the FL-NA. Note N is a positive integer. Then the solution can be obtained as

$$1.3739 \leq N \leq 20.6100 \ . \tag{28}$$

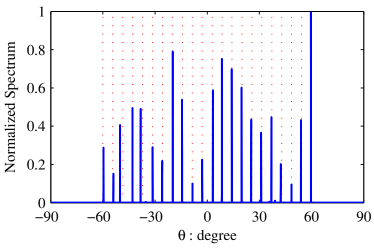
To ensure there are four sub-arrays in a FL-NA, N should be greater than 4. Therefore, for $N \leq 20$, our proposed structure can provide more DOFs than the FL-NA. The comparison of consecutive integers are listed in Table. 1. Furthermore, for 20 physical sensors with $(N_1, N_2, N_3) = (6, 7, 7)$ for our proposed structure, the number of virtual ULA sensors at the fourth-order difference co-array stage is 2223, which is sufficient for most applications. On the other hand, compared to the TL-NA, our extended structure always gives a significantly larger number of DOFs.



(a) DOA estimation results for the two-level nested array.



(b) DOA estimation results for the four-level nested array.



(c) DOA estimation results for our proposed structure.

Fig. 1. DOA estimation results for different array structures with $K = 22$.

3.4. Compressive sensing based DOA estimation employing the fourth-order difference co-array concept

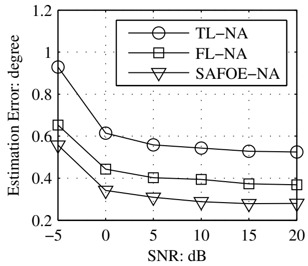
Apart from spatial smoothing based sub-space method, CS-based method can be applied to (8) for DOA estimation. With a search grid of K_g potential incident angles $\theta_{g,0}, \ldots, \theta_{g,K_g-1}$, a steering matrix can be constructed as $\mathbf{B_g} = \left[\mathbf{b}(\theta_{g,0}), \ldots, \mathbf{b}(\theta_{g,K_g-1})\right]$. Then a column vector $\mathbf{u_g}$ of size $K_g \times 1$ is constructed, with each entry representing a potential source signal at the corresponding incident angle. Then the CS-based DOA estimation employing the fourth-order difference co-array concept is formulated as

$$\min \quad \|\mathbf{u_g}\|_1 \quad \text{subject to} \quad \|\mathbf{y} - \mathbf{B_g}\mathbf{u_g}\|_2 \leq \varepsilon \ , \tag{29}$$

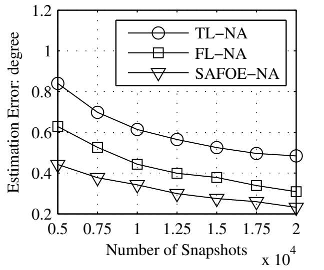
where ε is the allowable error bound, $\|\cdot\|_1$ is the ℓ_1 norm and $\|\cdot\|_2$ the ℓ_2 norm. The $K_g \times 1$ column vector $\mathbf{u_g}$ represents the DOA estimation results over K_g search grids. The optimization problem can be solved using CVX, a software package for specifying and solving convex problems [20, 21].

4. SIMULATION RESULTS

In our simulations, we consider examples with a small number of sensors: $(N_1, N_2) = (2, 3)$ for the standard TL-NA, $(N_1, N_2, N_3, N_4) = (1, 1, 1, 1)$ for the FL-NA with $\sum_{m=1}^{4} N_m + 1 = 5$ sensors, and $(N_1, N_2, N_3) = (1, 2, 2)$ for our proposed extended structure SAFOE-NA. The unit spacing $d = \lambda/2$, where λ is the signal wavelength. With a step size of 0.05°, a search grid of $K_g = 3601$ incident angles is generated within the full angle range from -90° to 90°. The allowable error bound ε is chosen to give the best results through trial-and-error for each scenario, and all the K source signals are uniformly distributed between -60° and 60°.



(a) RMSEs with different array structures versus input SNR.



(b) RMSEs with different array structures versus the number of snapshots.

Fig. 2. RMSE results with different array structures.

For the first set of simulations, the SNR is set to be 0 dB. To show the number of distinguishable sources, a sufficient number of snapshots for calculating the fourth-order cumulant matrix is used, fixed at 20000, and the number of sources $K = 22$. The DOA estimation results for difference array structures are shown in Fig. 1, where the dotted lines represent the actual incident angles of the impinging signals, whereas the solid lines represent the estimation results. With the same number of physical sensors, it is clear that both TL-NA and FL-NA have failed in resolving all these sources, while the proposed SAFOE-NA has achieved it successfully.

In the second set of simulations, we focus on the root mean square error (RMSE) results to compare the estimation accuracy of different array structures through Monte Carlo simulations of 500 trials. The number of sources K is 12. Fig. 2(a) gives the results with respect to a varied input SNR, where the number of snapshots is fixed at 10000. Clearly, the performance of our proposed array extension is the best among all the three structures, with that of the TL-NA being the worst. It is noted that the physical array aperture for the proposed structure is $23d$, while it is $15d$ for the FL-NA and $8d$ for the TL-NA. With the largest aperture, the proposed structure has consistently outperformed the other two existing ones.

Then, we fix the SNR to 0 dB, and the RMSE results versus different number of snapshots are shown in Fig. 2(b). We can see that, the larger the number of snapshots, the higher its estimation accuracy due to a better estimation of the statistics of the involved signals. Similarly, the performance of the proposed structure is still the best among all the three structures due to its larger aperture.

5. CONCLUSION

A sparse array extension based on the standard two-level nested array has been proposed to maximise the consecutive lags in the fourth-order difference co-array. After vectorizing the fourth-order cumulant matrix, a CS-based signal reconstruction method is then employed for effective DOA estimation. Given the same number of sensors, the number of consecutive lags and DOFs of the new structure is significantly larger than the existing two-level nested arrays; compared to the existing four-level nested arrays, when the sensor number is smaller than 21, the new structure also provides a larger number of consecutive lags (2223 for 20 sensors), which is sufficient for most applications. Moreover, it can be shown that among the three different structures, the proposed one has the largest aperture, leading to further improved performance.

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